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Solution of Dirac Equation in a Singularity-Free Kaluza–Klein Cosmological Model

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The singularity-free solution of $(4+D)$-dimensional Einstein field equations for the Kaluza–Klein cosmological model is obtained. Then the Dirac equations are solved in this model.

In the context of the unification of gravity with other fundamental forces of nature, Kaluza–Klein-type models (Kaluza, 1921; Klein, 1926*a*,*b*) are good candidates. In these theories (Duff *et al.*, 1986, and references therein) space-time is supposed to have the manifold structure $M^4 \times B^D$, where M^4 is the usual para-compact four-dimensional space-time (Minkowskian or non-Minkowskian) and B^D is the extra D-dimensional compact manifold (also called the internal manifold). The observable universe is 4-dimensional, hence B^D is supposed to have a very small size.

The coordinates of the $(4+D)$-dimensional manifold are separated into the coordinates x^μ ($\mu = 0, 1, 2, 3$) of M^4 plus the coordinates y^m ($m = 4, \ldots, D+3$) of B^D. Here, B^D is taken as T^D (D-dimensional torus), which is the product of D copies of circles. The line element is the generalized Robertson–Walker line element given as

$$ds^2 = dt^2 - \frac{r^2(t)}{\left(1 + \frac{k_3\sigma^2}{4}\right)^2}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - R^2(t)g_{mn}\, dy^m\, dy^n \quad (1a)$$

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where

$$\sigma^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \tag{1b}$$

$$g_{mn}\, dy^m\, dy^n = \rho_1^2\, d\theta_1^2 + \rho_2^2\, d\theta_2^2 + \cdots + \rho_D^2\, d\theta_D^2 \tag{1c}$$

($\rho_1, \ldots, \rho_D$ are the radii of the circles of T^D and $\theta_1, \ldots, \theta_D$ are angular coordinates), k_3 is the scalar curvature, having possible values $-1, 0, +1$ for open, flat, and closed models, respectively, and $r(t)$ and $R(t)$ are scale factors.

The $(4+D)$-dimensional Einstein field equations without cosmological constant are

$$G_{MN} = R_{MN} - \tfrac{1}{2} g_{MN} R = 8\pi G (T_{MN}^{(m)} + T_{MN}^{(\psi)}) \tag{2}$$

where G_{MN} is the Einstein tensor, R_{MN} is the Ricci tensor, G is the $(4+D)$-dimensional gravitational constant, $T_{MN}^{(m)}$ is the energy-momentum tensor for matter (which is supposed to be a perfect fluid) and $T^{(\psi)}$ is the energy-momentum tensor from the action for the Dirac spinor ψ given as

$$S^{(\psi)} = \frac{1}{2} \int dt\, d^3x\, d^D y\, \frac{r^3 R^D}{(1 + k_3 \sigma^2/4)^3} [g_D(y)]^{1/2} i\bar{\psi} \gamma^M D_M \psi \tag{3}$$

where D_M is the covariant derivative, γ^M ($M = 0, 1, 2, 3, \ldots, 3 + D$) are $(4+D)$-dimensional Dirac matrices, and $i = \sqrt{-1}$.

The line element on the 3-dimensional $t = \text{const}$ subspace N^3 of M^4 can be written from (1a) as

$$d\tilde{\sigma}^2 = \frac{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}{(1 + k_3 \sigma^2/4)^2} \tag{4}$$

A map f: $N^3 = \{(x^1, x^2, x^3)\} \to E^3 = \{(X^1, X^2, X^3)\}$, is defined such that

$$\frac{\partial X^i}{\partial x^i} = \left(1 + \frac{k_3 \sigma^2}{4}\right)^{-1} \quad \text{and} \quad \frac{\partial X^I}{\partial x^j} = 0 \quad (i \neq j) \tag{5}$$

The derivatives given in (5) form a Jacobian matrix which is nonsingular for $k_3 = +1$ only when $\sigma < \infty$. In the case $k_3 = -1$, these derivatives do not exist for $\sigma = 2$. Thus for a viable theory

$$\sigma \begin{cases} < \infty & \text{when} \quad k_3 = +1 \\ \neq 2 \quad \text{and} \quad < \infty & \text{when} \quad k_3 = -1 \end{cases} \tag{6}$$

The line element of E^3 can be written as

$$d\sigma_E^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \tag{7}$$

where

$$X^1 = \begin{cases} \left[\dfrac{4}{[4+(x^2)^2+(x^3)^2]^{1/2}} \tan^{-1} \dfrac{x^1}{[4+(x^2)^2+(x^3)^2]^{1/2}}\right]_{x^2=x^3=\text{const}} \\ \quad \text{when } k_3=+1 \\ \left[\dfrac{4}{[4-(x^2)^2-(x^3)^2]^{1/2}} \tanh^{-1} \dfrac{x^1}{[4-(x^2)^2-(x^3)^2]^{1/2}}\right]_{x^2=x^3=\text{const}} \\ \quad \text{when } k_3=-1 \\ x^1 \quad \text{when } k_3=0 \end{cases}$$

$$X^2 = \begin{cases} \left[\dfrac{4}{[4+(x^1)^2+(x^3)^2]^{1/2}} \tan^{-1} \dfrac{x^2}{[4+(x^1)^2+(x^3)^2]^{1/2}}\right]_{x^1=x^3=\text{const}} \\ \quad \text{when } k_3=+1 \\ \left[\dfrac{4}{[4-(x^1)^2-(x^3)^2]^{1/2}} \tanh^{-1} \dfrac{x^2}{[4-(x^1)^2-(x^3)^2]^{1/2}}\right]_{x^1=x^3=\text{const}} \\ \quad \text{when } k_3=-1 \\ x^2 \quad \text{when } k_3=0 \end{cases} \tag{8}$$

$$X^3 = \begin{cases} \left[\dfrac{4}{[4+(x^1)^2+(x^2)^2]^{1/2}} \tan^{-1} \dfrac{x^3}{[4+(x^1)^2+(x^2)^2]^{1/2}}\right]_{x^1=x^2=\text{const}} \\ \quad \text{when } k_3=+1 \\ \left[\dfrac{4}{[4-(x^1)^2-(x^2)^2]^{1/2}} \tanh^{-1} \dfrac{x^3}{[4-(x^1)^2-(x^2)^2]^{1/2}}\right]_{x^1=x^2=\text{const}} \\ \quad \text{when } k_3=-1 \\ x^3 \quad \text{when } k_3=0 \end{cases}$$

Thus, one finds that map f: $N^3 \to E^3$ is onto, but one-to-one only when $k_3=0$ or -1. From (5), it is clear that f is a c^1-function subject to the condition (6). It can be easily shown that f^{-1} is also a c^1-function, provided that conditions (6) are satisfied. So, in the case $k_3=0$ or -1, f is a c^1-diffeomorphism. One knows that a map which is a c^1-diffeomorphism is definitely a homeomorphism (Von Westenholz, 1978). Hence, N^3 can be uniformly deformed to E^3 with metric given by (7) subject to conditions (6) in the case $k_3=0$ or -1.

The action (3) for ψ is covariant as well as conformally invariant. Hence the action (3), in the background geometry given by (1), is equivalent to the action for ψ in the background geometry given by

$$ds^2 = dt^2 - r^2(t)[(dX^1)^2 + (dX^2)^2 + (dX^3)^2] - R^2(t)(\rho_1^2\, d\theta_1^2 + \cdots + \rho_D^2\, d\iota_D^2) \tag{9}$$

Hence

$$S_\psi = \int dt\, d^3X\, d^Dy\, r^3 R^D [g_D(y)]^{1/2} i\bar{\psi}\gamma^M D_M \psi \tag{10}$$

where $\gamma^M \equiv (\gamma^\mu, \gamma^m)$ can be written as

$$\gamma^0 = \tilde{\gamma}^0, \quad \gamma^1 = \frac{1}{r(t)}\tilde{\gamma}^1, \quad \gamma^2 = \frac{1}{r(t)}\tilde{\gamma}^2, \quad \gamma^3 = \frac{1}{r(t)}\tilde{\gamma}^3$$
$$\gamma^4 = \frac{\tilde{\gamma}^4}{R(t)}, \ldots, \quad \gamma^{3+D} = \frac{\tilde{\gamma}^{3+D}}{R(t)} \tag{11}$$

with

$$\tilde{\gamma}^\mu = \overset{*}{\gamma}{}^\mu \otimes I \tag{12}$$

$$\tilde{\gamma}^m = \overset{*}{\gamma}{}^5 \otimes \overset{*}{\gamma}{}^m \tag{13}$$

such that

$$\overset{*}{\gamma}{}^5 = i\overset{*}{\gamma}{}^0\overset{*}{\gamma}{}^1\overset{*}{\gamma}{}^2\overset{*}{\gamma}{}^3 (\overset{*}{\gamma}{}^5)^2 = 1$$

and $\overset{*}{\gamma}{}^m$ are Dirac matrices on B^D. The $\overset{*}{\gamma}{}^{\mu'}$ ($\mu = 0, 1, 2, 3$) are standard 4×4 Dirac matrices on 4-dimensional flat space-time. $\overset{*}{\gamma}{}^m$ and I (identity) are $2^{D/2} \times 2^{D/2}$ matrices (when D is even) and $2^{(D-1)/2} \times 2^{(D-1)/2}$ matrices (when D is odd). Here $\overset{*}{\gamma}{}^m$ are Dirac matrices on T^D, which is a manifold having $[U(1)]^D$ as its symmetry groups. So, $\overset{*}{\gamma}{}^m = I$ (Duff *et al.*, 1986; Gilbert and McClain, 1984; Wetterich, 1983).

The energy-momentum tensor for ψ can be derived from the action (3) or (10). For convenience, here it is derived from the action (10) as (Audretsch and Schafer, 1978)

$$T_{MN}^{(\psi)} = \frac{i}{2}[\bar{\psi}\gamma_{(M} D_{N)}\psi] \tag{14}$$

From the actions (3) and (10), it is obvious that ψ is a noninteracting field. Hence it will behave like a perfect fluid. So, we have from (14)

$$\epsilon^{(\psi)} = \langle 0|T_0^{(\psi)0}|0\rangle, \quad p^{(\psi)} = \langle 0|T_i^{(\psi)i}|0\rangle, \quad P^{(\psi)} = \langle 0|T_m^{(\psi)m}|0\rangle \tag{15}$$

Thus $\epsilon^{(\psi)}$, $p^{(\psi)}$, and $P^{(\psi)}$ are vacuum expectation values of $T_M^{(\psi)M}$. So, due to the spatial homogeneity of the model, these quantities have only time dependence.

Now, the total energy density, pressure on N^3, and pressure on T^D are written as $\epsilon = \epsilon^{(m)} + \epsilon^{(\psi)}$, $p = p^{(m)} + p^{(\psi)}$, and $P = P^{(m)} + P^{(\psi)}$, respectively, where $\epsilon^{(m)}$, $p^{(m)}$, and $P^{(m)}$ correspond to the perfect fluid (matter other than ψ).

The energy-momentum tensor T_{MN} for the perfect fluid (matter plus ψ) is written as (Gleiser and Taylor, 1985)

$$T_{MN} = (\epsilon + p)U_M U_N - (\delta p + \delta' P)g_{MN} \tag{16}$$

where

$$\delta = \begin{cases} 0 & \text{for} \quad M, N = m, n \\ 1 & \text{for} \quad M, N = \mu, \nu \end{cases} \quad \text{and} \quad \delta' = \begin{cases} 0 & \text{for} \quad M, N = \mu, \nu \\ 1 & \text{for} \quad M, N = m, n \end{cases} \tag{17}$$

and U^M is normalized as $U^M U_M = 1$ $(U^0 = +1,\ U^1 = U^2 = \cdots = U^{3+D} = 0)$.

Conservation of T^{MN} yields (Maeda, 1984)

$$\dot{\epsilon} + \epsilon\left(3\frac{\dot{r}}{r} + D\frac{\dot{R}}{R}\right) + 3p\frac{\dot{r}}{r} + DP\frac{\dot{R}}{R} = 0 \tag{18}$$

Einstein's equations can be written as

$$G_0^0 = \frac{6k_3 t_p^2}{r^2 l^2} + \left(3\frac{r'}{r} + D\frac{R'}{R}\right)^2 - 3\left(\frac{r'}{r}\right)^2 - D\left(\frac{R'}{R}\right)^2 = 8\pi\bar{G}t_p^2\epsilon \tag{19a}$$

$$\frac{2k_3 t_p^2}{r^2 l^2} + \frac{d}{d\tilde{t}}\left(\frac{r'}{r}\right) + \frac{r'}{r}\left(3\frac{r'}{r} + D\frac{R'}{R}\right) = -8\pi\bar{G}t_p^2 p \tag{19b}$$

$$\frac{d}{d\tilde{t}}\left(\frac{R'}{R}\right) + \frac{R'}{R}\left(\frac{3r'}{r} + D\frac{R'}{R}\right) = -8\pi\bar{G}t_p^2 P \tag{19c}$$

where prime denotes derivative with respect to the dimensionless parameter $\tilde{t} = t/tp$ and l is a constant of unit magnitude and the dimension of length.

Many solutions of (19) suffer from the disease of "crack of doom" singularity. Rosenbaum *et al.* (1987) have suggested diagrammatic solutions and have shown that, by an appropriate choice of parameters, the "crack of doom" singularity can be avoided. A realistic solution should be constant in the asymptotic limit $(t \to \infty)$. Gleiser and Taylor (1985) have derived a solution of this kind assuming suitable relations between ϵ, p, and P for the $D = 2$ case. Here, a simpler solution satisfying the above criteria for a realistic model is suggested.

It is assumed that

$$R^2(t) = f^2 + \frac{1}{r^2(t)} \tag{20}$$

where f is a constant.

Connecting (20) with (19b) and (19c), we have

$$\frac{2k_3 t_p^2}{r^2 l^2} + \frac{2f^2 r^2}{1+f^2r^2}\left(\frac{r'}{r}\right)^2 = -8\pi\bar{G}t_p^2[p^{(m)} + p^{(\psi)} + (P^{(m)} + P^{(\psi)})(1+f^2r^2)] \tag{21}$$

In order to solve equations (19), one needs equations of state. But no equation relating ε, p, and P is available. Nevertheless, one may impose the condition

$$p^{(m)} + p^{(\psi)} + (P^{(m)} + P^{(\psi)})(1+f^2r^2) = 0 \tag{22}$$

In (22), $p^{(\psi)}$ and $P^{(\psi)}$ can be calculated using (14) if one knows the solution of the Dirac equation ψ, but $p^{(m)}$ and $P^{(m)}$ are the pressures for an arbitrary perfect fluid. Condition (22) may be accepted provided that it is satisfactory from the physical point of view. From this condition, obviously $p^{(m)}$ and $P^{(m)}$ have time dependence only, due to spatial homogeneity. $p^{(m)}$ and $P^{(m)}$ can be positive or negative, depending on the signature of $p^{(\psi)}$ and $P^{(\psi)}$. In the case $p^{(\psi)} \ll p^{(m)}$ and $P^{(\psi)} \ll P^{(m)}$, (22) can be written as

$$p^m + P^{(m)}(1+f^2r^2) = 0$$

which is true at least for dust (known perfect fluid). Thus, the possibility of (22) can be accepted.

Under condition (22), equation (21) yields the exact solution

$$1 + f^2r^2 = \left(\frac{ft}{l} + |\alpha|\right)^2 \tag{23}$$

provided that $k_3 = -1$. If $k_3 = +1$, equation (21) yields a complex solution and the solution is constant if $k_3 = 0$. So, hereafter, only $k_3 = -1$ is considered.

Solutions (20) and (23) can be accepted if they satisfy the constraint (19a).

From conservation equations, $D_M G^{MN} = 0 = D_M T^{MN}$, one gets

$$\frac{\partial}{\partial t} G_0^0 = 0 = \frac{\partial}{\partial t} T_0^0 \tag{24}$$

In the model considered here, equation (24) implies that if (19a) is satisfied at one particular epoch, then it is always satisfied. So, one can choose an epoch $t = 0$.

Connecting equations (18) and (22) and integrating, one gets

$$\epsilon r^3 R^D = \epsilon_0 - \int_0^t \left[3(1+f^2 r^2)\frac{\dot r}{r} + D\frac{\dot R}{R}\right] P\, dt'$$

($\epsilon_0 = \text{const}$), which yields that

$$[\epsilon]_{t=0} = \left[\frac{\epsilon_0}{r^3 R^D}\right]_{t=0} \tag{25}$$

Now from (19a) at $t = 0$ one gets

$$-6t_p^2\alpha^2(\alpha^2-1) + (3\alpha^2 - D)^2 - 3\alpha^4 - D = \frac{8\pi \bar G t_p^2 \epsilon_0 (\alpha^2-1)^{(D+1)/2}}{f^{D-5}\alpha^{(D-1)}} \tag{26}$$

Equation (26) yields the condition under which solutions (20) and (23) satisfy the constraint equation (19a) at $t = 0$. Equation (26) can be written approximately as

$$\alpha^4 - 6D\alpha^2 + (D^2 - D) = 0 \tag{27}$$

neglecting the term on the r.h.s. of (26) as

$$\bar G = (2\pi)^D \rho_1 \cdots \rho_D G_N = (2\pi)^D \rho_1 \cdots \rho_D M_p^{-2}$$

Equation (27) yields

$$\alpha^2 = -\tfrac{1}{6}[3D \pm (3D^2 + 6D)^{1/2}] \tag{28}$$

From (28), one finds that $\alpha^2 = 1$ for $D = 1$ and $D = 6$ if only the negative sign is taken, otherwise $\alpha^2 > 1$ for every D. This result has the interesting feature that the solution $r(t)$ is singularity-free. As a result, the solution

$$R(t) = \left[f^2 + \frac{f^2}{(ft/l + |\alpha|)^2 - 1}\right]^{1/2} \tag{29}$$

also is singularity-free for $t \geq 0$.

Now we are in a position to solve the Dirac equation in the cosmological model discussed above. But before we do this, the action (10) should be dimensionally reduced to four-dimensions.

The metric g_{MN} provided by the line element (9) can be conformally transformed to

$$g_{MN} = R^2(t) g'_{MN} = R^2(t)\begin{pmatrix} \bar R^2 g_{\mu\nu} & 0 \\ 0 & -g_{mn}\end{pmatrix} \tag{30}$$

Under the transformation (30), the action (10) is

$$S^{(\psi)} = \int d\tau\, d^3X\, d^Dy \left(\frac{r}{R}\right)^3 [g_D(y)]^{1/2} i\bar\psi' \gamma'^M D'_M \psi' \tag{31a}$$

where

$$\tau = \int^t dt'/R(t') \tag{31b}$$

$$\psi = \bar{R}^{(3+D)/2}\psi' \tag{31c}$$

γ'^M are Dirac matrices with respect to the new metric g'_{MN}, and D'_M is the corresponding covariant derivative.

ψ' can be written in decomposed form as

$$\psi' = \sum_{n_1 \cdots n_D = -\infty}^{\infty} \exp\left(\frac{in_1 y_1}{\rho_1} + \cdots + \frac{in_D y_D}{\rho_D}\right)\phi^{(4)}_{(n)}(x) \tag{32}$$

where $\phi^{(4)}_{(n)}(x) = \phi^{(4)}_{n_1 n_2 \cdots n_D}(x)$ is a four-dimensional Dirac spinor. So the dimensionally reduced four-dimensional action for the Dirac spinor is

$$S^{(4)}_\phi = \sum_{(n)} \int d\tau\, d^3X \left(\frac{r}{R}\right)^3 \bar{\Phi}_{(n)}(i\gamma'^\mu D'_\mu + i\lambda_{(n)}\tilde{\gamma}^5)\phi_{(n)} \tag{33a}$$

where

$$\lambda_{(n)} = \frac{n_1}{\rho_1} + \cdots + \frac{n_D}{\rho_D} \tag{33b}$$

Here, the normalization condition

$$\int d^D y\, [g_D(y)]^{1/2} \chi_{(n')}\chi_{(n)} = \delta_{n'_1 n_1}\delta_{n'_2 n_2} \cdots \delta_{n'_D n_1} \tag{33c}$$

with

$$\chi_{(n)} = \exp\left(\frac{in_1 y_1}{\rho_1} + \cdots + \frac{in_D y_D}{\rho_D}\right)$$

has been used.

Under chiral rotation (Wetterich, 1983)

$$\phi_{(n)} \to \phi'_{(n)} = \exp\left(i\lambda_{(n)}\frac{\pi}{4}\tilde{\gamma}^5\right)\phi_{(n)}$$

the induced mass term in (33a) obtains the canonical form

$$\lambda_{(n)}\bar{\Phi}'_{(n)}\phi'_{(n)}$$

and the kinetic energy term is unaffected, so we have

$$S^{(4)}_\phi = \sum_{(n)} \int dt\, d^3X \left(\frac{r}{R}\right)^3 \bar{\Phi}'_{(n)}(i\gamma'^\mu D'_\mu + \lambda_{(n)})\phi'_{(n)} \tag{34}$$

The action (34) yields the Dirac equation for $\phi'_{(n)}$ as

$$i\gamma'^{\mu}D'_{\mu}\phi'_{(n)} + \lambda_{(n)}\phi'_{(n)} = 0 \tag{35}$$

The covariant derivative D'_{μ} is given as

$$D'_{\mu} = \partial_{\mu} - \Gamma_{\mu} \tag{36a}$$

where

$$\Gamma_{\mu} = \tfrac{1}{4}(\partial_{\mu}h^{\rho}_{a} + \Gamma^{\rho}_{\sigma\mu}h^{\sigma}_{a})g_{\nu\rho}h^{\nu}_{b}\tilde{\gamma}^{a}\tilde{\gamma}^{b} \tag{36b}$$

with the tetrad h^{ρ}_{a} defined as (Srivastava, 1989)

$$h^{\rho}_{a}h^{\sigma}_{b}g'_{\rho\sigma} = \eta_{ab} \tag{36c}$$

Hence

$$\Gamma_0 = 0, \qquad \Gamma_1 = -\frac{1}{2}\left(\frac{r''}{r} - \frac{R''}{R}\right)\tilde{\gamma}'\tilde{\gamma}^0$$
$$\Gamma_2 = -\frac{1}{2}\left(\frac{r''}{r} - \frac{R''}{R}\right)\tilde{\gamma}^2\tilde{\gamma}^0, \qquad \Gamma_3 = -\frac{1}{2}\left(\frac{r''}{r} - \frac{R''}{R}\right)\tilde{\gamma}^3\tilde{\gamma}^0 \tag{37}$$

where the double prime denotes derivative with respect to τ defined by equation (31b).

Now the Dirac equation (35) can be written as

$$\tilde{\gamma}^0\left[\partial_{\tau} + \frac{3}{2}\left(\frac{r''}{r} - \frac{R''}{R}\right)\right]\phi_{(n)} + \frac{R}{r}(\tilde{\gamma}^1\partial_1 + \tilde{\gamma}^2\partial_2 + \tilde{\gamma}^3\partial_3)\phi_{(n)} - i\lambda_{(n)}\phi_{(n)} = 0 \tag{38}$$

Setting

$$\phi_{(n)} = (2\pi r/R)^{-3/2}\exp(ik\cdot X)\begin{bmatrix} f_{\rm I}(k,\tau) \\ f_{\rm II}(k,\tau)\end{bmatrix} \tag{39}$$

and connecting it with equation (38), one gets the coupled equations

$$[\partial_{\tau} - i\lambda_{(n)}]f_{\rm I} + \frac{iR}{r}(k\cdot\sigma)f_{\rm II} = 0 \tag{40a}$$

$$[\partial_{\tau} + i\lambda_{(n)}]f_{\rm II} + \frac{iR}{r}(k\cdot\sigma)f_{\rm I} = 0 \tag{40b}$$

Equations (40) can be rewritten as

$$\left(\frac{\partial}{\partial\tilde{\tau}} - i\lambda_{(n)}\frac{r}{R}\right)f_{\rm I} + i(k-\sigma)f_{\rm II} = 0 \tag{41a}$$

$$\left(\frac{\partial}{\partial\tilde{\tau}} + i\lambda_{(n)}\frac{r}{R}\right)f_{\rm II} + i(k\cdot\sigma)f_{\rm I} = 0 \tag{41b}$$

where

$$\tilde{\tau} = \int^{t} \frac{dt'}{r(t')} \tag{41c}$$

From equation (41a)

$$f_{\text{II}} = \frac{i(k \cdot \sigma)}{k^2}\left(\frac{\partial}{\partial \tilde{\tau}} - i\lambda_{(n)}\frac{r}{R}\right)f_{\text{I}} \tag{42a}$$

and connecting it with (41b), we have

$$\left(\frac{\partial}{\partial \tilde{\tau}} + i\lambda_{(n)}\frac{r}{R}\right)\left(\frac{\partial}{\partial \tilde{\tau}} - i\lambda_{(n)}\frac{r}{R}\right)f_{\text{I}} + k^2 f_{\text{I}} = 0$$

which can be rewritten as

$$\frac{\partial^2}{\partial \tilde{\tau}^2}f_{\text{I}} + \left[k^2 + \frac{m^2 r^2}{R^2} - im\frac{\partial}{\partial \tilde{\tau}}\left(\frac{r}{R}\right)\right]f_{\text{I}} = 0 \tag{42b}$$

where

$$\tilde{\tau} = l \cosh^{-1}\left(\frac{ft}{l} + |\alpha|\right)$$

Using the solutions (20) and (23), we find equation (42b) reduces to

$$\frac{d^2 f_{\text{I}}}{d\tilde{\tau}^2} + \left\{k^2 + m^2 f^{-4} \tanh^2\frac{\tilde{\tau}}{l}\sinh^2\frac{\tilde{\tau}}{l} - i\lambda_{(n)}\frac{f^2}{l}\sinh\left(\frac{\tilde{\tau}}{l}\right)\left[2 - \tanh^2\left(\frac{\tilde{\tau}}{l}\right)\right]\right\}f_I = 0 \tag{43}$$

For very small $\tilde{\tau}$, equation (43) is approximated as

$$\frac{d^2 f_{\text{I}}}{d\tilde{\tau}^2} + (k^2 - 2i\lambda_{(n)} f^2 l^{-2}\tilde{\tau})f_{\text{I}} = 0 \tag{44}$$

which yields the solution (Murphy, 1960)

$$f_{\text{I}} = z^{1/2}[N_1 J_{1/3}(\tfrac{2}{3}z^{3/2}) + N_2 J_{-1/3}(\tfrac{2}{3}z^{3/2})] \tag{45a}$$

where

$$z = \frac{[k^4 + 4\lambda_{(n)}^2 (f^2/l^2)\tilde{\tau}^2]^{1/2}}{(2\lambda_{(n)} f^2 l^2)^{2/3}} \exp\left[i\left(\frac{\pi}{3} - \theta\right)\right] \tag{45b}$$

with

$$\theta = \tan^{-1}(2\lambda_{(3)}\tilde{\tau}/f^2 l^2 k^2) \tag{45c}$$

and $J_p(x)$ is the Bessel function.

Connecting equations (45a) with equation (42a), we find

$$\begin{aligned} f_{\rm II} = {} & (2\lambda_{(n)}\bar{f}^2\bar{l}^2)^{1/3}\frac{e^{i\pi/3}}{2R^2}(k\cdot\sigma)z^{-1/2} \\ & \times\{N_3[J_{1/3}(\tfrac{2}{3}z^{3/2}) + z^{3/2}J_{-2/3}(\tfrac{2}{3}z^{3/2}) - z^{3/2}J_{4/3}(\tfrac{2}{3}z^{3/2})] \\ & + N_4[J_{-1/3}(\tfrac{2}{3}z^{3/2}) + z^{3/2}J_{-4/3}(\tfrac{2}{3}z^{3/2}) - z^{3/2}J_{2/3}(\tfrac{2}{3}z^{3/2})]\} \end{aligned} \tag{45d}$$

Corresponding to $f_{\rm I}$ and $f_{\rm II}$ given by equations (45a) and (45d), $\phi_{(n)}$ can be written as

$$\begin{aligned} \phi_{(n){\rm I}s} = {} & \left(2\pi\frac{r}{R}\right)^{-3/2}\exp(ik\cdot X) \\ & \times z^{1/2}[N_1 u_s J_{1/3}(\tfrac{2}{3}z^{3/2}) + N_2\hat{u}_s J_{-1/3}(\tfrac{2}{3}z^{3/2})] \end{aligned} \tag{46a}$$

and

$$\begin{aligned} \phi_{(n){\rm II}s} = {} & (2\lambda_{(n)}\bar{f}^2\bar{l}^2)^{1/3}\frac{e^{i\pi/3}}{2k^2}(k\cdot\sigma)z^{1/2}\left(2\pi\frac{r}{R}\right)^{-3/2}\exp(ik\cdot X) \\ & \times\{N_3 u_s[J_{1/3}(\tfrac{2}{3}z^{3/2}) + z^{3/2}J_{-2/3}(\tfrac{2}{3}z^{3/2}) - z^{3/2}J_{4/3}(\tfrac{2}{3}z^{3/2})] \\ & + N_4\hat{u}_s[J_{-1/3}(\tfrac{2}{3}z^{3/2}) + z^{3/2}J_{-4/3}(\tfrac{2}{3}z^{3/2}) - z^{3/2}J_{2/3}(\tfrac{2}{3}z^{3/2})]\} \end{aligned} \tag{46b}$$

where z is defined by equations (45b) and (45c) and for spin quantum number $s = \pm 1$, u_s ($\hat{u}_s$) are given as column matrices

$$\begin{aligned} & \hat{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{u}_{-1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ & u_1 = \begin{pmatrix} 0 \\ 0 \\ -k_3 \\ -k_1 - ik_2 \end{pmatrix}, \qquad \hat{u}_{-1} = \begin{pmatrix} 0 \\ 1 \\ -k_1 + ik_2 \\ k_3 \end{pmatrix} \end{aligned} \tag{47}$$

The normalization constants N_i can be evaluated using the condition that the norm defined as

$$(\phi^k, \phi^{k'}) = \int_{t=\text{const}} \sqrt{-g_4}\, d^3x\, \bar{\phi}_s^k \tilde{\gamma}^0 \phi_{s'}^{k'} \tag{48}$$

approaches $(2\pi)^{-3}\delta_{ss'}\delta^3(k-k')$ in the flat space limit (Srivastava, 1989). So,

$$\begin{aligned}
N_1 &= [kJ_{1/3}(\tfrac{2}{3}z_1^{3/2})]^{-1} \\
N_2 &= [J_{-1/3}(\tfrac{2}{3}z_1^{3/2})]^{-1} \\
N_3 &= 2[2\lambda_{(n)}\tilde{f}^2\tilde{l}^2]^{-1/3}z_1^{1/2}[J_{1/3}(\tfrac{2}{3}z_1^{3/2}) \\
&\quad + z_1^{3/2}J_{-2/3}(\tfrac{2}{3}z_1^{3/2}) - z_1^{3/2}J_{4/3}(\tfrac{2}{3}z_1^{3/2})]^{-1} \\
N_4 &= 2[2\lambda_{(n)}\tilde{f}^2\tilde{l}^2]^{-1/3}kz_1^{1/2}[J_{-1/3}(\tfrac{2}{3}z_1^{3/2}) \\
&\quad + z_1^{3/2}J_{-4/3}(\tfrac{2}{3}z_1^{3/2}) - z_1^{3/2}J_{2/3}(\tfrac{2}{3}z_1^{3/2})]^{-1}
\end{aligned} \tag{49}$$

where

$$z_1 = \frac{[k^4 + 4\lambda_{(n)}^2\tilde{f}^2(\cosh^{-1}\alpha)^2]^{1/2}}{(2\lambda_{(n)}\tilde{f}^2\tilde{l}^2)^{2/3}} \exp\left[i\left(\frac{\pi}{3} - \theta_1\right)\right]$$

with

$$\theta_1 = \tan^{-1}[2\lambda_{(n)}\tilde{f}^2\tilde{l}^1\tilde{k}^2 \cosh^{-1}\alpha]$$

For large $\tilde{\tau}$, equation (43) is approximated as

$$\frac{d^2f_1}{d\tilde{\tau}^2} + \left[k^2 + m^2\tilde{f}^4e^{2\tilde{\tau}/l} - i\lambda_{(n)}\frac{\tilde{f}^2}{l^2}e^{\tilde{\tau}/l}\right]f_1 = 0 \tag{50}$$

which is integrated to

$$\begin{aligned}
f_1 = \exp\left(-\frac{\tilde{\tau}}{2l} + \frac{k\tilde{\tau}}{l} + Le^{\tilde{\tau}/l}\right)&\left[c_1\, {}_1F_1\left(\frac{2KL+b}{2L}, 2K, -2Le^{\tilde{\tau}/l}\right)\right. \\
&\left. + c_2(-2le^{-\tilde{\tau}/l})^{1-2K}\, {}_1F_1\left(1 + \frac{2KL+b}{2L} - 2K, 2-2K, -2Le^{\tilde{\tau}/l}\right)\right]
\end{aligned} \tag{51a}$$

where

$$K^2 - K + l^2k^2 = 0 \tag{51b}$$

$$L^2 = -m^2\tilde{f}^4l^2 \tag{51c}$$

$$b = -i\lambda_{(n)}\tilde{f}^2l \tag{51d}$$

From equations (42) and (51)

$$f_{\mathrm{II}} = \frac{i(k \cdot \sigma)}{k^2} \exp\left(-\frac{\tilde{\tau}}{2l} + \frac{k\tilde{\tau}}{l} + Le^{\tilde{\tau}/l} \right) [c_3 Y_1(\tilde{\tau}) + c_4 Y_2(\tilde{\tau})] \tag{52a}$$

where

$$Y_1(\tilde{\tau}) = \left(\frac{-1 + 2k + 2L}{2l} e^{\tilde{\tau}/l} - \frac{i\lambda_{(n)}}{2f^2} e^{\tilde{\tau}/l} \right) {}_1F_1\left(\frac{2KL + b}{2L}, 2K, -2Le^{\tilde{\tau}/l} \right)$$
$$+ \frac{2KL + b}{2KL} {}_1F_1\left(\frac{2KL + b}{2L} + 1, 1 + 2K, -2Le^{\tilde{\tau}/l} \right) \tag{52b}$$

and

$$Y_2(\tilde{\tau}) = \left(\frac{-1 + 2k + 2Le^{\tilde{\tau}/l}}{2l} - i\frac{\lambda_{(n)}}{2f^2} e^{\tilde{\tau}/l} \right)$$
$$\times {}_1F_1\left(1 + \frac{2KL + b}{2L} - 2K, 2 - 2K, -2Le^{\tilde{\tau}/l} \right)$$
$$+ {}_1F_1\left(2 + \frac{2KL + b}{2L} - 2K, 3 - 2K, -2Le^{\tilde{\tau}/l} \right) \tag{52c}$$

with K, L, b as defined in equation (51).

Corresponding to f_{I} and f_{II} given by equations (51) and (52), the components of $\phi_{(n)}$ can be written as

$$\phi_{(n)\mathrm{I},s} = \left(\frac{2\pi r}{R} \right)^{-3/2} \exp\left(ik \cdot X - \frac{\tilde{\tau}}{2l} + \frac{k\tilde{\tau}}{l} + Le^{\tilde{\tau}/l} \right)$$
$$\times \left[c_1 u_s\, {}_1F_1\left(\frac{2KL + b}{2L}, 2K, -2Le^{\tilde{\tau}/l} \right) \right.$$
$$\left. + c_2 \hat{u}_s (-2le^{\tilde{\tau}/l})^{1-2K}\, {}_1F_1\left(1 + \frac{2KL + b}{2L} - 2K, 2 - 2K, -2Le^{\tilde{\tau}/l} \right) \right] \tag{53a}$$

and

$$\phi_{(n)\mathrm{II},s} = \left(\frac{2\pi r}{R} \right)^{-3/2} \exp\left(ik \cdot X - \frac{\tilde{\tau}}{2l} + \frac{k\tilde{\tau}}{l} + Le^{\tilde{\tau}/l} \right) \frac{i(k \cdot \sigma)}{k^2}$$
$$\times [c_3 u_s Y_1(\tilde{\tau}) + c_4 \hat{u}_s Y_2(\tilde{\tau})] \tag{53b}$$

The normalization constants in equations (53) are calculated as

$$c_1 = \left[k\,{}_1F_1\left(\frac{2KL+b}{2L}, 2K, -2Le^{\tilde{\tau}_0/l}\right)\right]^{-1} \exp\left(\frac{\tilde{\tau}}{2l} - \frac{k\tilde{\tau}_0}{l} - Le^{\tilde{\tau}_0/l}\right)$$

$$c_2 = (-2le^{\tilde{\tau}_0/l})^{2K-1}\left[{}_1F_1\left(1 + \frac{2KL+b}{2L} - 2K, 2-2K, -2Le^{\tilde{\tau}_0/l}\right)\right]^{-1}$$

$$\times \exp\left(\frac{\tilde{\tau}_0}{2l} - \frac{k\tilde{\tau}_0}{l} - Le^{\tilde{\tau}_0/l}\right)$$

$$c_3 = [kY_1(\tilde{\tau}_0)]^{-1} \exp\left(\frac{\tilde{\tau}_0}{2l} - \frac{k\tilde{\tau}_0}{l} - Le^{\tilde{\tau}_0/l}\right)$$

$$c_4 = [Y_2(\tilde{\tau}_0)]^{-1} \exp\left(\frac{\tilde{\tau}_0}{2l} - \frac{k\tilde{\tau}_0}{l} - Le^{\tilde{\tau}_0/l}\right)$$

where $\tilde{\tau} = l \cosh^{-1} \alpha$.

REFERENCES

Audretsch, J., and Schafer, G. (1978). *Journal of Physics A*, **11**, 1583.

Duff, M. J., Nilsson, B. E. W., and Pope, C. N. (1986). *Physics Reports*, **130**, 1–142.

Gilbert, G., and McClain, B. (1984). *Nuclear Physics B*, **244**, 173.

Gleiser, M., and Taylor, J. G. (1985). *Physical Review D*, **31**, 1904.

Kaluza, T. (1921). *Sitzungberichte der Preussichen Akademie der Wissenschaften Berlin, Mathematik Physik*, **1921**, 966.

Klein, O. (1926*a*). *Zeitschrift für Physik*, **37**, 875.

Klein, O. (1926*b*). *Nature*, **118**, 516.

Maeda, K. (1984). *Physical Review D*, **30**, 2482.

Murphy, G. M. (1960). *Ordinary Differential Equations*, Von Nostrand, Englewood Cliffs, New Jersey.

Rosenbaum, M., Ryan, M., Urrutia, L., and Matzner, R. (1987). *Physical Review D*, **36**, 1032.

Srivastava, S. K. (1989). *Journal of Mathematical Physics*, **30**, 2838.

Von Westenholz, C. (1978). *Differential Forms in Mathematical Physics*, North-Holland, Amsterdam.

Wetterich, C. (1983). *Nuclear Physics B*, **222**, 20.